Visualizing elements of Sha[3] in genus 2 jacobians

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Joint work with Nils Bruin

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We want to describe elements of $\text{Ш}(E/k)$ explicitly.

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Equivalently, σ lies in the kernel of $H^1(k, E) \to H^1(k, A)$.

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Theorem (Bruin & D.)

If $\sigma \in Ш(E/k)$ has order 3, then σ is visible in the jacobian of a curve of genus 2.

Mazur's criterion in tems of Galois cohomology

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$$\text{So } \Delta(\overline{k}) = \{(P, \lambda(P)) : P \in E[3](\overline{k})\}.$$

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Fisher: $\mathrm{Jac}(C')[3] \simeq \mathrm{Jac}(C)[3]$ anti-isometrically.
Some work: $\mathrm{Flex}(C),\ \mathrm{Flex}(C')$ have isomorphic $\mathrm{Gal}(\overline{k}/k)$-action.

The genus 2 curve

Make sure that $E := \mathrm{Jac}(C)$ and $E' := \mathrm{Jac}(C')$ are not isogenous, then $A := (E \times E')/\Delta \simeq \mathrm{Jac}(X)$ for a genus 2 Curve X/k.

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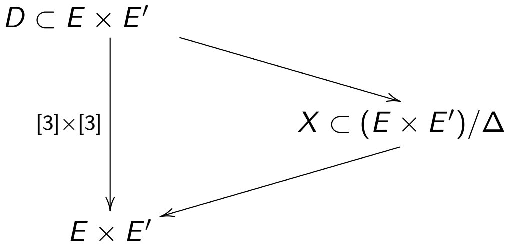
- Frey and Kani: X is the image of D in $(E \times E')/\Delta$.

Constructing the genus 2 curve

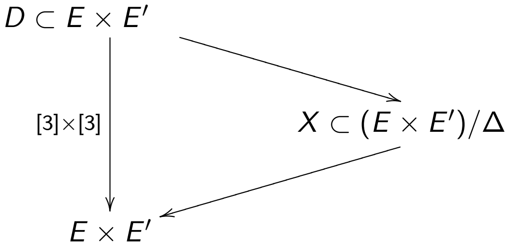


Constructing the genus 2 curve

$$\begin{array}{ccc} D \subset E \times E' & \searrow & \\ {\scriptstyle [3]\times[3]}\big\downarrow & & X \subset (E \times E')/\Delta \\ E \times E' & \swarrow & \end{array}$$

- The map $[3] \times [3]$ is much more accessible.

Constructing the genus 2 curve



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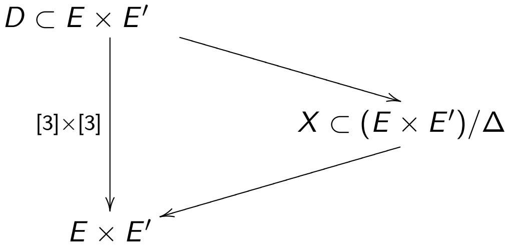
Constructing the genus 2 curve



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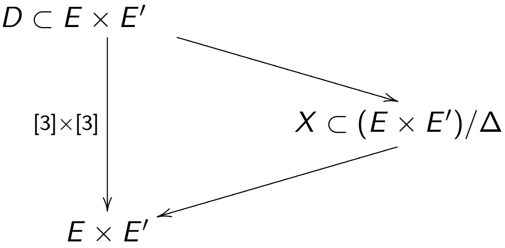
Constructing the genus 2 curve



- The map $[3] \times [3]$ is much more accessible.
- The subgroup of $E[3] \times E'[3]$ under which D is invariant is Δ.
- Hence a model for X can be found on $E \times E'$ as $([3] \times [3])(D)$.
- This image can easily be computed by interpolation.

Fin

Detailed examples:

Fin

Detailed examples: See the proceedings . . .

Fin

Detailed examples: See the proceedings . . .

Thank you for your attention.