Applications of Explicit Coleman Integration for Hyperelliptic Curves

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Introduction to Coleman Integration

Notation:

- C hyperelliptic curve over an unramified extension k of $\mathbb{Q}_p$ with p a prime of good ordinary reduction
- Points P, Q, R on C
- Differential forms ω, ω' of the second kind on C
- Differential forms $\omega_0, \ldots, \omega_{2g-1}$ a basis for $H^1_{dR}(C)$, where $\omega_i = \frac{x^i dx}{2y}$

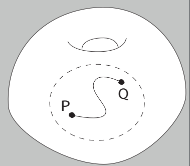
Coleman constructed a definite integral with the following properties:

1. Linearity: $\int_P^Q (\alpha\omega + \beta\omega') = \alpha \int_P^Q \omega + \beta \int_P^Q \omega'$.
2. Additivity: $\int_P^R \omega = \int_P^Q \omega + \int_Q^R \omega$.
3. Change of variables: If C' is another curve and $\phi : C \to C'$ a rigid analytic map between wide opens then $\int_P^Q \phi^*\omega = \int_{\phi(P)}^{\phi(Q)} \omega$.
4. Fundamental theorem of calculus: $\int_P^Q df = f(Q) - f(P)$.

"Tiny" Integrals

Suppose $P, Q \in C(\mathbb{C}_p)$ are in the same residue disc. We compute $\int_P^Q \omega_i$ locally:

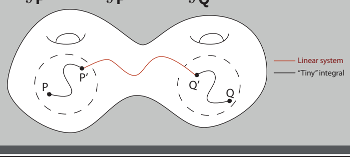
1. Construct an interpolation $x(t), y(t)$ from P to Q.
2. Formally integrate the power series in t: $\int_P^Q \omega_i = \int_P^Q x^i \frac{dx}{2y} = \int_0^1 \frac{x(t)^i}{2y(t)} \frac{dx(t)}{dt} dt$.



Integrals via Kedlaya's algorithm

If P, Q are in different residue discs, we use Frobenius ϕ to construct $\int_P^Q \omega_i$:

1. Find Teichmüller points P', Q' in the discs of P, Q.
2. Compute the tiny integrals $\int_P^{P'} \omega_i, \int_{Q'}^{Q} \omega_i$.
3. Calculate the action of Frobenius on each basis element $\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j$.
4. Change of variables gives $\sum_{j=0}^{2g-1} (M - I)_{ij} \int_{P'}^{Q'} \omega_j = f_i(P') - f_i(Q')$, and solving the linear system gives the integrals $\int_{P'}^{Q'} \omega_i$.
5. Correct endpoints to recover $\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_{P'}^{Q'} \omega_i + \int_{Q'}^{Q} \omega_i$.



Application: Coleman-Gross height pairing

The Coleman-Gross height pairing is a symmetric bilinear pairing

$$h : \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \to \mathbb{Q}_p,$$

which can be written as a sum of local height pairings $h = \sum_v h_v$ over all finite places v of the number field K.

Local height above p

Let $D_1, D_2 \in \mathrm{Div}^0(C)$ have disjoint support and ω_{D_1} be a normalized differential associated to D_1. The local height pairing at v above p is given by

$$h_v(D_1, D_2) = \mathrm{tr}_{k/\mathbb{Q}_p}\left(\int_{D_2} \omega_{D_1}\right).$$

To construct ω_{D_1}:

- Choose a differential ω with $\mathrm{Res}(\omega) = D_1$.
- Fix a splitting
$$H^1_{dR}(C/k) = H^{1,0}_{dR}(C/k) \oplus W,$$
where W is the unit root subspace for the action of Frobenius.
- Via the canonical homomorphism $\Psi : T(k)/T_l(k) \longrightarrow H^1_{dR}(C/k)$, compute $\Psi(\omega) = \eta + \Psi(\omega_{D_1})$, for η holomorphic. Then $\omega_{D_1} := \omega - \eta$.

Coleman integration: meromorphic differential

Let ϕ be a p-power lift of Frobenius and set $\alpha := \phi^*\omega - p\omega$. Then for β a differential with residue divisor $D_2 = (R) - (S)$, we compute

$$\int_{D_2} \omega = \int_S^R \omega = \frac{1}{1-p}\left(\Psi(\alpha) \cup \Psi(\beta) + \sum \mathrm{Res}\left(\alpha \int \beta\right)\right) - \frac{1}{1-p}\left(\int_{\phi(S)}^{S} \omega + \int_R^{\phi(R)} \omega\right).$$

Example: global p-adic heights for genus 1

Example: Let C be the elliptic curve $y^2 = x^3 - 5x$, with $Q = (-1, 2), Q' = (-1, -2), R = (5, 10), R' = (5, -10)$, so that $(Q) - (Q') = (R) - (R') = (\frac{9}{4}, -\frac{3}{8}) =: P$.

We compute the 13-adic height of P:

- Above 13, the local height $h_{13}((Q) - (Q'), (R) - (R'))$ is $2 \cdot 13 + 6 \cdot 13^2 + 13^3 + 5 \cdot 13^4 + O(13^5)$.
- Away from 13, the only nontrivial contribution is $2 \log 3$ (by work of Müller).
- So the global 13-adic height is $12 \cdot 13 + 4 \cdot 13^2 + 10 \cdot 13^3 + 9 \cdot 13^4 + O(13^5)$.

We compare this to Harvey's implementation of Mazur-Stein-Tate in Sage:

```
sage:  C = EllipticCurve([-5,0])
sage:  f = C.padic_height(13)
sage:  f(C(9/4,-3/8)) + O(13^5)
12*13 + 4*13^2 + 10*13^3 + 9*13^4 + O(13^5)
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Application: Kim's nonabelian Chabauty method

Kim's nonabelian Chabauty method allows us to recover integral points on elliptic curves:

Theorem:
Let $\mathcal{C}/\mathbb{Z}$ be the minimal regular model of an elliptic curve $C/\mathbb{Q}$ of analytic rank 1 with Tamagawa numbers all 1. Let $\mathcal{X} = \mathcal{C} - \{\infty\}$ and $\omega_0 = \frac{dx}{2y}, \omega_1 = \frac{xdx}{2y}$. Taking a tangential base point b at ∞, let $\log_{\omega_0}(z) = \int_b^z \omega_0, D_2(z) = \int_b^z \omega_0\omega_1$. Suppose y is a point of infinite order in $\mathcal{C}(\mathbb{Z})$. Then $\mathcal{X}(\mathbb{Z}) \subset \mathcal{C}(\mathbb{Z}_p)$ is in the zero set of

$$f(z) := \log^2_{\omega_0}(y) D_2(z) - \log^2_{\omega_0}(z) D_2(y).$$

Computing $D_2(z)$: Double Coleman integrals

We take as our normalization $\int_P^Q \omega_i\omega_j := \int_P^Q \omega_i(R) \int_P^R \omega_j$.

A straightforward generalization of single Coleman integration yields the following techniques:

- "Tiny" double integration (points P, Q in the same residue disc)
  - Compute local coordinates $x(t), y(t)$ at P, and let $R = (a + x(Q), \sqrt{f(a + x(Q))})$.
  - Write $\int_P^Q \omega_i\omega_j = \int_0^{x(Q)-x(P)} \left(\int_0^a \frac{x(t)^j dx(t)}{2y(t)}\right) \frac{x(R(a))^i}{2y(R(a))} \frac{dx(R(a))}{da}$.
- Linking integrals between non-Weierstrass points via Frobenius:
  - Compute Teichmüller points P', Q' in the discs of P, Q.
  - Use Frobenius to calculate $\int_{P'}^{Q'} \omega_i\omega_k$.
  - Recover the double integral: $\int_P^Q \omega_i\omega_k = \int_{P'}^{Q'} \omega_i\omega_k - \int_{P'}^{P} \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{P'}^{P} \omega_k\right) - \left(\int_Q^{Q'} \omega_i\right)\left(\int_{P'}^{Q'} \omega_k\right) + \int_{Q'}^{Q} \omega_i\omega_k$.

Example: integral points

Let $E : y^2 = x^3 - 16x + 16$ (which has minimal model 37a1). Given two integral points x, y of infinite order, a third point z occurs in the zero set of the function

$$\left(\left(\int_b^z \omega_0\right)^2 - \left(\int_b^x \omega_0\right)^2\right) \frac{\int_x^y \omega_0\omega_1 + \int_x^y \omega_0 \int_b^x \omega_1}{(\int_b^y \omega_0)^2 - (\int_b^x \omega_0)^2} - \left(\int_x^z \omega_0\omega_1 + \int_x^z \omega_0 \int_b^x \omega_1\right).$$

Indeed, fixing $x = (0, 4), y = (4, 4)$ on E, we may recover $z = (-4, -4), (8, -20), (24, 116)$.

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