Hyperbolic tessellations associated to Bianchi groups

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Overview

The space of positive definite n-ary Hermitian forms over a number field F forms an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoï polyhedron.

We investigate this space in the case where $n = 2$ and F is an imaginary quadratic field, yielding tessellations of hyperbolic 3-space. As an application, we use the tessellation to get information about the arithmetic group $\mathrm{GL}_2(\mathcal{O}_F)$.

Applications and related work

1. Group presentations
2. Group (co)-homology
3. Hecke operators acting on Bianchi modular forms.

Grunewald, Elstrodt, Mennicke, Mendoza, Schwermer, Vogtmann, Flöge, Cremona and students, Swan, Riley, Rahm-Fuchs, Sengün.

Review of the rational case

$n = 2$, $F = \mathbb{Q}$

Every binary quadratic form can be represented by a symmetric 2×2 real matrix. Let C be the 3-dimensional open cone of positive definite quadratic forms.

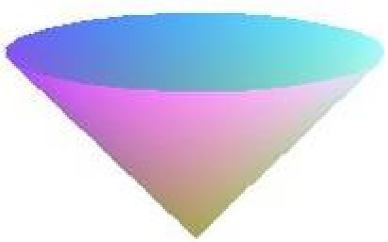

Figure: Cone of positive definite forms.

Review of the rational case

Voronoï polyhedron

The *Voronoï polyhedron* Π is the closed convex hull in $\bar{C}$ of

$$\{vv^t \;:\; v \in \mathbb{Z}^2 \setminus 0\}.$$

Review of the rational case

Tessellation by ideal triangles

Π is an infinite polyhedron whose faces are triangles.

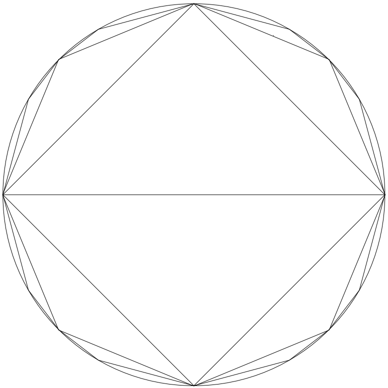

Figure: Trace $= 1$ slice

Review of the rational case

Tessellation by ideal triangles

This tessellation descends to give tessellation of $\mathfrak{h}$ by ideal triangles.

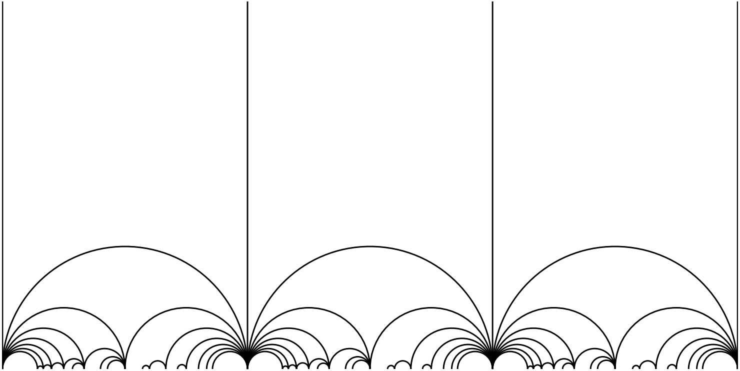

Figure: Tessellation of $\mathfrak{h}$ by ideal triangles.

Hermitian forms over F

$n = 2$, $F =$ imaginary quadratic field

Let V be the 4 dimensional real vector space of Hermitian 2×2 matrices.

1. The positive definite Hermitian matrices forms an open cone $C \subset V$.
2. $\mathrm{GL}_2(\mathcal{O}_F)$ acts on C by

$$\gamma \cdot A = \gamma A \gamma^*.$$

Ideal hyperbolic polytopes

We can identify C/H with 3-dimensional hyperbolic space $\mathbb{H}^3$.
$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$ is the analogue of $\mathfrak{h}$, the complex upper half-plane.

The *Voronoï polyhedron* Π is the unbounded polyhedron gotten by taking the convex hull in $\bar{C}$ of

$$\{vv^* \ : \ v \in \mathcal{O}_F^2 \setminus 0\}.$$

Ideal hyperbolic polytopes

Cusps

The points $vv^* \in \overline{C}$ correspond to ideal points (cusps), which are the points $F \cup \infty$. The facets of Π descend to a tessellation of $\mathbb{H}^3$ by ideal polytopes.

Hermitian forms over F

For $A \in C$, The *minimum of A* is

$$m(A) = \inf_{v \in \mathcal{O}_F^2 \setminus \{0\}} v^* A v.$$

A vector $v \in \mathcal{O}_F^2$ is *minimal vector* for A if $v^* A v = m(A)$. The set of minimal vectors for A is denoted $M(A)$.

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A Hermitian form over F is *perfect* if it is uniquely determined by $M(A)$ and $m(A)$.

Voronoï polyhedron

Let I be a facet of the Voronoï polyhedron with vertices V_I. There exists a unique perfect form ϕ_I with $m(\phi_I) = 1$ such that

$$\{vv^* \ : \ v \in M(\phi_I)\} = V_I.$$

Voronoï polyhedron

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We search for a perfect form by looking in the 1-parameter family of forms

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Once an initial form is found, the $\mathrm{GL}_2(\mathcal{O}_F)$-classes are found by "flipping across facets".

Ideal polytope data

Table: Combinatorial types of ideal polytopes that occur in this range.

polytope	F-vector	picture
tetrahedron	$[4, 6, 4]$	
octahedron	$[6, 12, 8]$	
cuboctahedron	$[12, 24, 14]$	
triangular prism	$[6, 9, 5]$	
hexagonal cap	$[9, 15, 8]$	
square pyramid	$[5, 8, 5]$	
truncated tetrahedron	$[12, 18, 8]$	
triangular dipyramid	$[5, 9, 6]$	

Ideal polytope data

Table: Voronoï ideal polytopes for class number 1.

h_F	d								
1	-1	0	1	0	0	0	0	0	0
1	-2	0	0	1	0	0	0	0	0
1	-3	1	0	0	0	0	0	0	0
1	-7	0	0	0	1	0	0	0	0
1	-11	0	0	0	0	0	0	1	0
1	-19	0	0	1	1	0	0	0	0
1	-43	0	0	0	2	1	0	1	0
1	-67	0	1	0	2	1	2	1	0
1	-163	11	0	1	8	2	3	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 2.

h_F	d								
2	-5	0	0	0	2	0	0	0	0
2	-6	0	0	0	0	1	0	1	0
2	-10	0	1	0	1	0	2	0	0
2	-13	1	0	0	3	1	1	0	0
2	-15	1	1	0	0	0	0	0	0
2	-22	5	0	1	4	0	2	0	0
2	-35	3	4	0	1	0	2	0	0
2	-37	10	0	0	8	1	8	0	0
2	-51	1	0	1	2	1	0	1	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 2.

h_F	d								
2	-58	47	0	0	7	2	6	0	0
2	-91	5	1	0	5	0	3	0	0
2	-115	3	1	0	5	2	4	0	0
2	-123	1	1	1	6	3	3	1	0
2	-187	18	1	1	4	1	9	1	0
2	-235	13	1	0	12	4	11	0	0
2	-267	24	1	1	13	5	10	1	0
2	-403	66	1	0	16	2	20	0	2
2	-427	65	2	0	19	4	24	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 3.

h_F	d								
3	-23	0	1	0	1	0	1	0	0
3	-31	0	0	0	3	0	1	0	0
3	-59	0	1	1	3	0	2	0	0
3	-83	6	0	0	2	2	1	1	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 4.

h_F	d								
4	-14	5	0	0	3	0	1	0	0
4	-17	5	0	0	2	1	3	1	0
4	-21	8	2	0	2	1	4	0	0
4	-30	6	0	0	6	4	4	0	0
4	-33	9	0	1	8	1	6	1	0
4	-34	20	0	0	3	1	6	1	0
4	-39	1	0	0	3	1	1	0	0
4	-46	32	1	0	5	0	9	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 4.

h_F	d								
4	-55	5	1	0	2	0	2	0	0
4	-57	33	1	0	10	3	14	2	0
4	-73	57	1	1	13	1	14	0	2
4	-78	69	1	0	11	4	18	0	0
4	-82	92	0	0	8	3	11	1	0
4	-85	56	0	0	17	0	28	0	0
4	-93	79	1	0	20	7	21	0	0
4	-97	95	0	1	19	3	19	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 5 and 6.

h_F	d								
5	−47	5	0	0	1	1	2	0	0
5	−79	9	0	0	5	0	4	0	0
6	−26	18	1	0	2	1	4	0	0
6	−29	15	0	0	6	0	6	0	0
6	−38	33	1	0	2	1	6	1	0
6	−53	45	0	0	7	2	13	0	0
6	−61	41	1	0	11	1	16	0	0
6	−87	6	0	0	6	2	3	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 7 and 8.

h_F	d								
7	−71	7	1	0	4	0	4	0	0
8	−41	31	0	1	9	0	8	0	0
8	−62	81	0	0	7	2	7	0	0
8	−65	69	2	0	9	0	19	0	0
8	−66	67	1	1	9	4	12	1	0
8	−69	51	2	0	15	2	21	0	0
8	−77	81	1	0	9	2	26	0	0
8	−94	125	1	0	10	2	17	0	0
8	−95	12	0	0	4	0	9	0	0

Ideal polytope data

Table: Voronoï ideal polytopes for class number 10 and 12.

h_F	d								
10	-74	105	1	0	9	1	12	0	0
10	-86	130	0	0	9	1	18	1	0
12	-89	136	0	0	14	1	21	1	0

Group presentation from topology

A general result of Macbeath and Weil gives the following.

Theorem

Suppose a space X is acted upon by a group of homeomorphisms Γ. Let $U \subset X$ be an open subset, and let $\Sigma \subset \Gamma$ denote the set

$$\Sigma = \{g \in \Gamma : g \cdot U \cap U \neq \emptyset\}.$$

Let $W \subset \Sigma \times \Sigma$ be the set

$$W = \{(g, h) : U \cap g \cdot U \cap gh \cdot U \neq \emptyset\}.$$

Let $R \subset F(\Sigma)$ denote the subgroup generated by $x_g x_h x_{(gh)^{-1}}$ for $(g, h) \in W$. For X, U sufficiently nice,

$$\Gamma \simeq F(\Sigma)/R.$$

Group presentation from topology

How nice is *nice*?

1. $\Gamma \cdot U = X$.
2. $\pi_0(X) = 0$. (X is connected.)
3. $\pi_1(X) = 0$. (X is simply-connected.)
4. $\pi_0(U) = 0$. (U is connected.)

Example: $F = \mathbb{Q}(\sqrt{-14})$

Theorem

The following is a presentation of $\mathrm{GL}_2(\mathbb{Z}[\sqrt{-14}])$*:*

$$\mathrm{GL}_2(\mathcal{O}_F) = \langle g_1, \cdots, g_8 : R_1 = \cdots = R_{22} = 1 \rangle, \quad \textit{where}$$

$$\begin{array}{llll}
R_1 = g_7^2, & R_2 = g_8^2, & R_3 = g_6^2, & R_4 = g_3^2, \\
R_5 = g_4^2, & R_6 = g_2^2, & R_7 = g_5^4, & R_8 = (g_2 g_1^{-1})^2, \\
R_9 = (g_4 g_1)^2, & R_{10} = g_5^{-1} g_1^{-3} g_5^{-1}, & R_{11} = (g_7 g_5^{-2})^2, & R_{12} = (g_8 g_5^{-2})^2, \\
R_{13} = (g_6 g_5^{-2})^2, & R_{14} = (g_4 g_5^{-2})^2, & R_{15} = (g_3 g_5^{-2})^2, & R_{16} = (g_6 g_1^{-1} g_5^{-1})^2,
\end{array}$$

$$R_{17} = (g_3 g_5^{-1} g_3 g_1 g_2)^2, \quad R_{18} = (g_3 g_7 g_1 g_8 g_1^{-1})^2, \quad R_{19} = g_4 g_5 g_4 g_1^{-1} g_5 g_1,$$

Example: $F = \mathbb{Q}(\sqrt{-14})$

Theorem continued

$$R_{20} = g_8 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_3 g_7 g_1 g_8 g_3 g_5 g_7 g_5^{-1},$$
$$R_{21} = g_1 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7,$$
$$R_{22} = g_6 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_6 g_1^{-1} g_7 g_3 g_1 g_3 g_5 g_7 g_5.$$

Example: $F = \mathbb{Q}(\sqrt{-14})$

Theorem continued

$$\begin{aligned}
R_{20} &= g_8 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_3 g_7 g_1 g_8 g_3 g_5 g_7 g_5^{-1}, \\
R_{21} &= g_1 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7, \\
R_{22} &= g_6 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_6 g_1^{-1} g_7 g_3 g_1 g_3 g_5 g_7 g_5.
\end{aligned}$$

Corollary

$\mathrm{GL}_2(\mathbb{Z}[\sqrt{14}])$ *has no torsion-free quotients.*

Thank you.